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Edge and modular edge irregularity strength of some path related graphs

A.I. Almazaydeh

Abstract. For a simple, connected and undirected graph $G(V, E)$ the mapping $\phi : V(G) \longrightarrow \{1, 2, \ldots, k\}$ that is defined from the vertex set $V(G)$ of the graph G to positive integers is called a *vertex k-labelling.* Let x and y be two vertices in $V(G)$, the *weight of the edge xy* -denoted by $w(xy)$- is defined to be the sum of the label of the vertex x and the label of the vertex y. That is $w_\phi(xy) = \phi(x) + \phi(y)$.

An *edge irregular k-labelling* of a graph G is defined to be a vertex k-labelling in which the weights of two distinct edges are not equal. The *edge irregularity strength*, denoted by $es(G)$, is an edge irregular k-labelling where k is the smallest such that the weights of the edges are distinct. If, by using some k-labelling where k is as above, the weight of each edge is divided by modulo the total number of the edges of the graph G, and the answers are all distinct, then that k-labelling is called a *modular edge irregularity strength.*

Haryeni et al. in [8] found that the edge irregularity strength of fan graphs F_n where $n \in \{2, 3, 4, 5, 6\}$ is $n + 1$. In this paper, we generalise this result for $n = 2, 3, 4, \ldots$. Also we state the edge irregularity strength and modular edge irregularity strength for some lollipop graphs.

AMS Subject Classification (2020): 05C78, 05C38

Keywords: Simple graph, k-labelling, irregularity strength, modular irregularity, fan graph, complete graph, lollipop graph

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1. Introduction

Consider a graph G with vertex set $V(G)$ and edge set $E(G)$, which is simple, connected and undirected. A concept of edge irregular labelling was given by Chartrand et al. in [6] which is improved later by Ahmad et. al. in [1]. The concern of edge irregularity strength gain momentum recently where many researches were done to calculate the edge irregularity strength of different types of graphs, we mention for example, P_n, star graphs of the form $K_{1,n}$, cartesian product of two paths P_n, P_m and double star graph where their precise values can be found in [1]. The value of edge irregularity strength of toeplitz graphs was given by Ahmad et al. in [2], and of triangular grid graphs in [14].

Recently, the edge irregularity strength of disjoint union of n-copies of graphs, the complete graphs and the wheel graphs have been found in [4]. Many more graphs were their edge irregularity strength are stated in [3], [9] and [13].

The number of vertices of a graph G is called the *order* of G, and the cardinality of the edges of a graph G is called the *size* of G. For a vertex $v \in V$, the number of edges that is connected to v is called the *degree of* v. We denote $\Delta(G)$ to be $max\{deg(v) : v \in V\}$. An important result worth mentioning is the lower bound of the edge irregularity strength of any graph G that is stated in the following Lemma.

Lemma 1.1 [1]. *For a simple graph G of size m and maximum degree $\Delta(G)$, $es(G) \geq max\{\lceil \frac{m+1}{2} \rceil,\ \Delta(G)\}$.*

2. Fan graphs

A *fan* graph $F_n, n \geq 2$, is a graph obtained by joining all vertices of a path P_n on n vertices to a further vertex, called the *centre*, which we denote in this paper by u. So, F_n is isomorphic tho the join of the path P_n and K_1, denoted by $P_n + K_1$. This type of graphs is known as the *join of two graphs* and is defined as follows:

Consider two disjoint graphs G and H. The *join* $G + H$ is the union of G and H with the vertex set $V(G) \bigcup V(H)$ and the edge set $E(G) \bigcup E(H)$

such that the edges in $G+H$ are the ones that joining every vertex of G to every vertex of H. For more you can see [16], and for its edge irregularity strength see [3].

Remark 2.1. It is worth mentioning that F_n is neither isomorphic to the complete bipartite graph $K_{1,n}$ nor to the star graph but the last two are isomorphic as it is witnessed in Figure 1. The edge irregularity strength of $K_{1,n}$ is given in [11], where the edge irregularity strength of the complete bipartite graph $K_{m,n}$ is found in [15].

On the other hand, note that the *symmetric product* of P_n and the complement of the complete graph $\overline{K_m}$, which is denoted by $P_n \oplus \overline{K_m}$ and is defined to be the graph with the vertex set $V(P_n) \times V(\overline{K_m})$, and $\{(u,v)(u',v') : uu' \in E(P_n)\}$ is the edge set. The edge irregularity strength of the symmetric product $P_n \oplus \overline{K_2}$ is stated in [9].

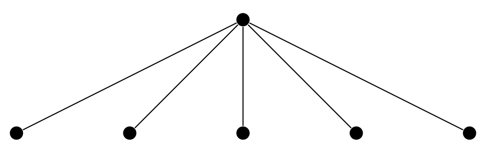

Figure 1: Star graph, $K_{1,n}$

For any fan graph F_n,$|V(F_n)| = n+1$, where those vertices are the vertices $v_1, v_2, \ldots, v_n$ of the path P_n and the centre u, and $|E(F_n)| = 2n-1$ which are the edges uv_i where $1 \leq i \leq n$ and the edges v_iv_j where $1 \leq i \leq j \leq n$.

The lower and upper bounds for any F_n have been stated in the following lemmas.

Lemma 2.2 [8]**.** *Let $F_n, n \geq 2$, be a fan graph of order $n+1$. Then $es(F_n) \geq n+1$.* □

Lemma 2.3 [8]**.** *Let $G = P_n + K_1$, $n \geq 2$. Then $n+1 \leq es(G) \leq$*

$n + \lfloor \frac{n}{2} \rfloor$. □

In addition, the edge irregularity strength of F_n, $n = 2, 3, 4, 5, 6$ is given in [8] by $es(F_n) = n + 1$. Now we generalise this to any n, but we first present how to label the vertices of some fan graphs. Note that the first five fan graphs are labelled in [8]. However, they can also be labelled by the procedure we follow in labelling fan graphs in this paper to agree with the proof of Theorem 2.4 below.

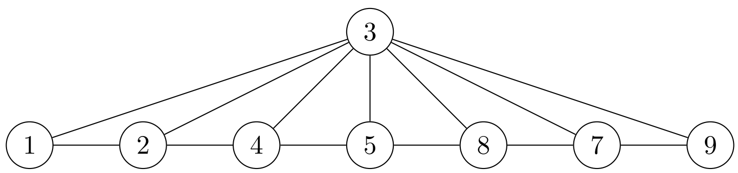

Figure 2: $es(F_7) = 9$

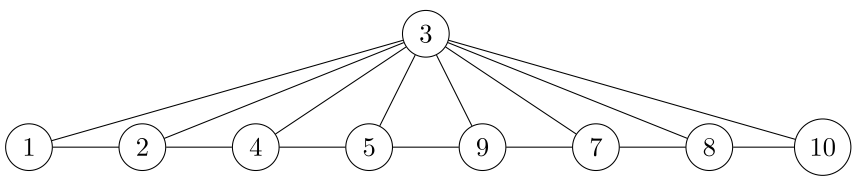

Figure 3: $es(F_8) = 10$

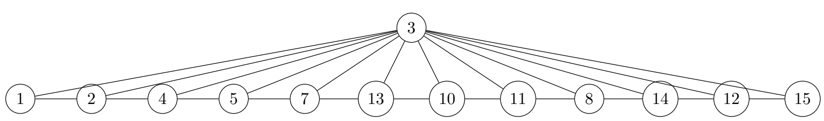

Figure 4: $es(F_{11}) = 14$

Theorem 2.4. *Let F_n be a fan graph of order $n+1$, $n \geq 2$. Then* $es(F_n) = \begin{cases} 3, & \text{if } n = 2 \\ n + 1 + \lfloor \frac{n-3}{4} \rfloor, & \text{if } n \geq 3 \end{cases}$

Proof. Let F_n be a fan graph that has $n + 1$ vertices. Firstly, if $n = 2$ it is clearly that the edge irregularity strength is 3. We see by Lemma 2.2

and Lemma 2.3 that the claimed edge irregularity strength of a fan graph satisfies the lower and upper bounds such that $es(F_n) = n + 1 + \lfloor \frac{n-3}{4} \rfloor \geq n + 1$, and $n + 1 + \lfloor \frac{n-3}{4} \rfloor \leq n + \lfloor \frac{n}{2} \rfloor$ for $n \geq 3$.

To prove the claimed assertion we show that there is an edge irregular $n + 1 + \lfloor \frac{n-3}{4} \rfloor$-labelling. Define a mapping $\phi : V(F_n) \longrightarrow \{1, 2, 3, \ldots, k\}$ where $k = n + 1 + \lfloor \frac{n-3}{4} \rfloor$ to be a vertex labelling such that $\phi(u) = 3$ (the centre), $\phi(v_1) = 1$, $\phi(v_2) = 2$, $\phi(v_3) = 4$, $\phi(v_4) = 5$ and $\phi(v_i) \in \{6, 7, 8, \ldots, k\}$ where $i = 5, 6, 7, \ldots, n$.

Clearly ϕ is a one-to-one mapping because for any two vertices $x \neq y$ we can see that $\phi(x) \neq \phi(y)$. Then the edge weights will be divided into two sets:

- the set E_1 consists of edge weights of the vertices uv_i, i.e. the edges joining the centre u with the vertices v_i in the path P_n such that $w_\phi(uv_i) \in \{4, 5, 7, \ldots, k+3\}$ where the edge weight 6 does not appear because $\phi(u) = 3$ implies that $\phi(v_i) \neq 3$ for all $i = 1, \ldots, n$, and $k+3$ is the largest possible edge weight, because it is the largest sum of $\phi(u)$ (which is 3) and $\phi(v_n)$ (which is k), hence the largest edge weight is $n + 1 + \lfloor \frac{n-3}{4} \rfloor + 3 = n + 4 + \lfloor \frac{n-3}{4} \rfloor$.

  It is readily seen that all the labels of the vertices are, at most, of the value $n + 1 + \lfloor \frac{n-3}{4} \rfloor$, and its the smallest value that produces distinct edge weights for all the distinct vertices.

- the second set of edge weights is E_2, the set of edge weights of the vertices $v_i v_j$, $1 \leq i \leq j \leq n$ such that $w_\phi(v_i v_j) \in \{3, 6, 9, 11, 12, \ldots, 2k-1\}$ where $2k - 1$ is the largest possible edge weight because it is the addition of $\phi(v_n)$ and the edge weight of its predecessor v_{n-1}, so it is $k + k - 1 = 2k - 1$.

Now if we emerge E_1 and E_2 we get the edge set of the graph F_n, namely $E = \{3, 4, 5, 6, \ldots, k+3, \ldots, 2k-1\}$ where all the edge weights are distinct, so it can be observed that ϕ is an edge irregular k-labelling. $\square$

3. Lollipop graphs

Another type of graphs is obtained by gluing a path with a complete graph by an edge, this type is called a *lollipop* graph.

Definition 3.1. A *lollipop* graph consists of a complete graph K_n joined from one of its vertices by a bridge to a path graph P_t. This graph is denoted by $L_{n,t}$.

In this context, we draw the edge connecting K_n with P_t by a dashed line. It is clear that $|V(L_{n,t})| = n + t$. As the size of the complete graph is $\frac{1}{2}n(n-1)$, and of the path P_t is $t-1$ and there is one more bridge joining K_n with P_t, then the size of $L_{n,t} = \frac{1}{2}n(n-1) + t$.

For the lollipop graphs, there are some of the calculated parameters, for instance *distance irregularity strength* in [11] and *total vertex irregularity strength* in [12]. We now state the edge irregularity strength for some lollipop graphs.

As the complete graph K_n is a part of the lollipop graph, we use $es(K_n)$ in our labelling of the vertices of $L_{n,t}$. For this reason we include the following theorem, and to keep this paper self contained the proof of this theorem will be listed.

Theorem 3.2. *Consider the complete graph $G = K_n$ of order $n \geq 3$. Then $es(G) = \left\lfloor \frac{\varphi^{n+1}}{\sqrt{5}} + \frac{1}{2} \right\rfloor$, where $\varphi = \frac{1+\sqrt{5}}{2}$.*

Proof. Let $G = K_n$ be the complete graph of order $n \geq 3$, with the vertices set V, and and the edges set is E. Define the vertex labeling map $\phi : V \to \{1, 2, \cdots, k\}$ such as

$$\phi(v_1) = 1,\ \phi(v_2) = 2$$

and

$$\phi(v_m) = \phi(v_{m-1}) + \phi(v_{m-2})$$

for all $m = 3, 4, \cdots, n$.

Then, the labelings of the vertices $v_1, v_2, v_3, \ldots, v_n$ is $\phi(v_1), \phi(v_2), \phi(v_3), \cdots, \phi(v_n)$ which form the Fibonacci sequence of the terms 1, 2, 5, 8, $\cdots$, $k = \left\lfloor \frac{\varphi^{n+1}}{\sqrt{5}} + \frac{1}{2} \right\rfloor$ with corresponding edge weights $3, 4, 5, \cdots, \phi(v_{n-1}) + \phi(v_n)$ which are all distinct.

Therefore, ϕ is an edge irregular k-labelling of G, and if $es(G) \leq k$ then the used labelling map is not one-to-one, for which it will be not irregular (every vertices are adjacent). Therefore $es(G) \geq k$.

Using the fact that $es(G) \leq \mathcal{F}_n$ (see [1]) , where $\mathcal{F}_n$ is the Fibonacci number, it results that $es(G) \leq \mathcal{F}_n = k$. Hence, the claim follows. □

Theorem 3.3. *Let $G = L_{3,t}$ be the lollipop graph of $3 + t$ vertices. Then $es(G) = \lceil \frac{3+t}{2} \rceil + 1$.*

Proof. Consider the lollipop graph $L_{3,t}$ with $3 + t$ vertices, which are v_1, v_2, v_3 of K_3 and the vertices $u_1, u_2, \ldots, u_t$ of the path P_t, with size $3+t$. So $\Delta(G) = 3$, and using Lemma 1.1, $es(G) \geq \; max\{\lceil \frac{4+t}{2} \rceil, 3\} = \lceil \frac{4+t}{2} \rceil$ so the lower bound for $es(G)$ is 3 which agrees with the least value of $es(G)$.

To prove the claim of this theorem, we need to define an edge irregular $\lceil \frac{3+t}{2} \rceil + 1$-labelling, and this can be done as follows:

Define the map $\phi : V(G) \longrightarrow \{1, 2, 3, \ldots, \lceil \frac{3+t}{2} \rceil + 1\}$ such that $\phi(v_1) = 1$, $\phi(v_2) = 2$, $\phi(v_3) = 3$ and $\phi(u_i) = 3 + \lfloor \frac{i}{2} \rfloor$ for $i = 1, 2, \ldots, t$. This map assigns every vertex of G to a distinct value. The set of the edges is $\{v_1v_2, v_1v_3, v_2v_3, v_3u_1, u_1u_2, \ldots, u_{t-1}u_t\}$ and their weights are $\{3, 4, 5, 6, 7, \ldots, 6 + \lfloor \frac{t-1}{2} \rfloor + \lfloor \frac{t}{2} \rfloor\}$ which forms a set of different integers. All this ensures that ϕ is the needed bijection which completes the proof. □

Remark 3.4. Note that $L_{3,t}$ is isomorphic to $(3,t) - kite$ graph (defined below), where its edge irregularity strength is stated in [9], and it can be seen that our result agrees with it.

Definition 3.5. A graph that consists of a cycle on m vertices, $m \geq 3$, and a path of t vertices such that the cycle and the path are connected by a bridge is called the *(m,t)-kite* graph.

Example 3.6. Here is an example on labelling $L_{3,4}$

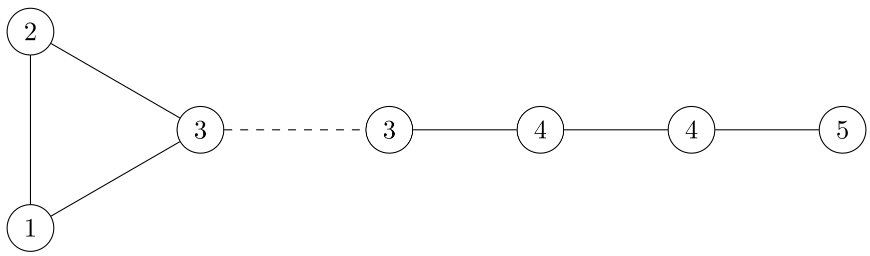

Figure 5: $es(L_{3,4}) = 5$

Theorem 3.7. *Let $G = L_{4,t}$ be the lollipop graph of $4+t$ vertices. Then $es(L_{4,t}) = 5 + \left\lfloor \frac{t}{2} \right\rfloor$.*

Proof. Consider the graph $G = L_{4,t}$ which has a set of $4+t$ vertices, namely, $\{v_1, v_2, v_3, v_4, u_1, u_2, \ldots, u_t\}$ and its edges set $\{v_1v_2, v_1v_3, v_1v_4, v_2v_3, v_2v_4, v_3v_4, v_4u_1, u_1u_2 \ldots, u_{t-1}u_t\}$.

Note that $\Delta(G) = 4$, so by Lemma 1.1, $es(G) \geq \; max\{\left\lceil \frac{5+t}{2} \right\rceil, 4\} = \left\lceil \frac{5+t}{2} \right\rceil$ for $t \geq 2$. If $t = 1$ then $max\{\left\lceil \frac{5+t}{2} \right\rceil, 4\} = 4$ which still less than the least value $es(L_{4,1}) = 5$.

To prove the assertion, we define an edge irregular k-labelling, where $k = 5 + \left\lfloor \frac{t}{2} \right\rfloor$, such that $\phi : V(G) \longrightarrow \{1, 2, \ldots, k\}$. The labelling of the vertices is done by letting $\phi(v_1) = 1, \phi(v_2) = 2, \phi(v_3) = 3, \phi(v_4) = 5, \phi(u_1) = 4$ and $\phi(u_i) = 5 + \left\lfloor \frac{i}{2} \right\rfloor, \; i \geq 2$.

Clearly, this map is one-to-one. For the edge set $\{v_1v_2, v_1v_3, v_1v_4, v_2v_3, v_2v_4, v_3v_4, v_4u_1, u_1u_2 \dots, u_{t-1}u_t\}$, the corresponding wights are $\{3, 4, 6, 5, 7, 8, 9, \dots, 10 + \lfloor \frac{t-1}{2} \rfloor + \lfloor \frac{t}{2} \rfloor\}$ respectively, and they all are distinct. Thus ϕ is the required irregular k-labelling. □

Example 3.8. In this example we show the edge irregularity strength of $L_{4,5}$.

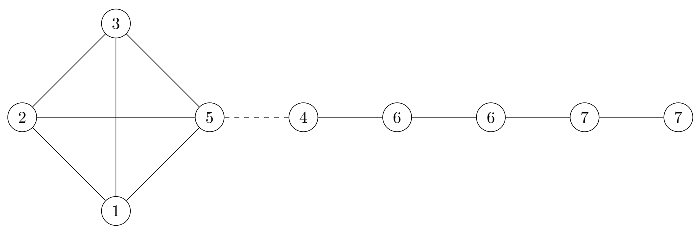

Figure 6: $es(L_{4,5}) = 7$

Theorem 3.9. *Let $G = L_{5,t}$ be the lollipop graph of $5 + t$ vertices. Then $es(G) = 8$ for $t = 1$, and $es(G) = 8 + \lfloor \frac{t+2}{2} \rfloor$ for $t \geq 2$.*

Proof. In this graph, $\Delta(G) = 5$ and $|V(G)| = 5 + t$, so

$$es(G) \geq \; max\{\left\lceil \frac{6+t}{2} \right\rceil, 5\} = \begin{cases} 5, & \text{if } t = 1, 2 \\ \lceil \frac{6+t}{2} \rceil & \text{if } t \geq 3 \end{cases}$$

and this inequality agrees with the $es(L_{5,1})$ that has the least value equals to 8.

First, for $t = 1$ we have $es(L_{5,1}) = 8$ where its vertices can be labelled as in Figure 7 below.

For $t \geq 2$, We define an edge irregular k-labelling ϕ, with $k = 9 + \lfloor \frac{t+2}{2} \rfloor$ such that $\phi : V(G) \longrightarrow \{1, 2, \dots, k\}$, and the labelling can be done as the following: the 5 vertices of the complete graph K_5 are v_1, v_2, v_3, v_4, v_5 and their labels are $\phi(v_1) = 1, \phi(v_2) = 2, \phi(v_3) = 3, \phi(v_4) = 5, \phi(v_5) = 8$ and

$$\phi(u_i) = \begin{cases} 4 + \left\lfloor \frac{i}{2} \right\rfloor, & \text{if i is odd} \\ 8 + \frac{i}{2}, & \text{if i is even} \end{cases}$$

So ϕ maps each vertex to only one value.

On the other hand, the set of edges is $\{v_1v_2, v_1v_3, v_1v_4, v_1v_5, v_2v_3, v_2v_4, v_2v_5, v_3v_4, v_3v_5, v_4v_5, v_5u_1, u_1u_2, \ldots, u_{t-1}u_t\}$, and using the map ϕ their weights are $\{3, 4, 6, 9, 5, 7, 10, 8, 11, 13, 12, 14 \ldots, 18 + \left\lfloor \frac{t+1}{2} \right\rfloor + \left\lfloor \frac{t+2}{2} \right\rfloor\}$ respectively, which is equivalent to the set $\{2, 3, 4, \ldots, 18 + \left\lfloor \frac{t+1}{2} \right\rfloor + \left\lfloor \frac{t+2}{2} \right\rfloor\}$, and this is a set of distinct integers, which completes the proof. □

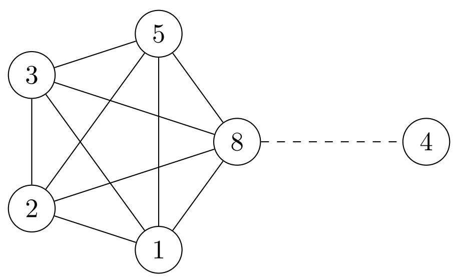

Figure 7: $es(L_{5,1}) = 8$

Example 3.10. We present the edge irregularity strength of $L_{5,6}$.

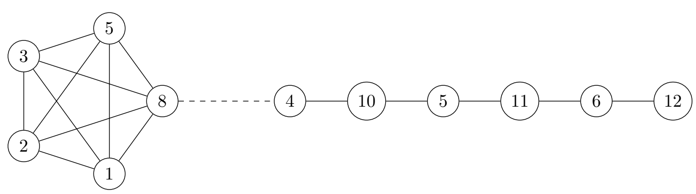

Figure 8: $es(L_{5,6}) = 12$

Theorem 3.11. *Let $G = L_{6,t}$ be the lollipop graph of $6 + t$ vertices. Then*

$$es(G) = \begin{cases} 13, & \text{if } 1 \leq t \leq 6 \\ 13 + \left\lfloor \frac{t-6}{2} \right\rfloor, & \text{if t} > 6 \end{cases}$$

Proof. Let $L_{6,t}$ be a lollipop graph. According to Lemma 1.1,

$$es(G) \geq max\{\left\lceil \frac{7+t}{2} \right\rceil, 6\} = \begin{cases} 6, & \text{if } t = 1,2,3,4,5 \\ \lceil \frac{7+t}{2} \rceil, & \text{if } t \geq 6 \end{cases}$$

and these values are less than the $es(L_{6,t})$ which is 13.

To proceed we consider two cases:

(i) if $1 \leq t \leq 6$, we define the map $\phi : V(G) \longrightarrow \{1, 2, 3, \ldots, 13\}$ which assigns the vertices $\{v_1, v_2, v_3, v_4, v_5, v_6\}$ of the complete graph K_6 to $\{1, 2, 3, 5, 8, 13\}$ respectively, and the vertices of the path P_t, where t is at most is 6, such that $\{4, 8, 11, 9, 13, 10\}$ for $u_1, u_2, u_3, u_4, u_5, u_6$ where P_1 needs only u_1, P_2 needs only u_1 and u_2, and so on for P_3 to P_6.

It is trivial that all the labels are distinct, and therefore the edge weights are also distinct and they form the set of consecutive integers $\{3, 4, 5, \ldots, 23\}$ in some order. Thus, in this case ϕ is an edge irregular 13-labelling. Here is the figure for this case.

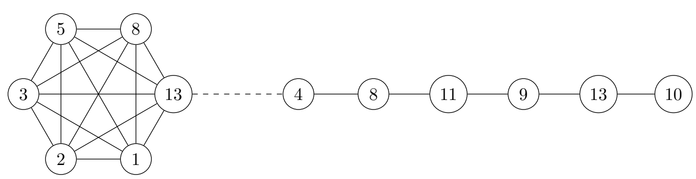

Figure 9: $es(L_{6,6}) = 13$

(ii) if $t > 6$, we again define a map $\phi : V(G) \longrightarrow \{1, 2, \ldots, k\}$ where $V(G) = \{v_1, \ldots, v_6, u_1, \ldots, u_t\}$, such that $\{\phi(v_1), \ldots, \phi(v_6)\} = \{1, 2, 3, 5, 8, 13\}$ respectively, fix $\phi(u_1) = 4$, $\phi(u_3) = 11$, and

$$\phi(u_i) = \begin{cases} 8 + \frac{i-2}{2}, & \text{if i is even} \\ 13 + \frac{i-5}{2}, & \text{if i is odd and } i \geq 5 \end{cases}$$

All these labels are distinct. (Note that this labelling is also works for the first case above). The set of the edges is $\{v_1v_2, v_1v_3, v_1v_4, v_1v_5, v_1v_6, v_2v_3, v_2v_4, v_2v_5, v_2v_6, v_3v_4, v_3v_5, v_3v_6, v_4v_5, v_4v_6, v_5v_6, v_6u_1, u_1u_2, u_2u_3, \ldots, u_{t-1}u_t\}$ and its corresponding set of weights is $\{3, 4, 6, 9, 14, 5, 7, 10, 15, 8, 11, 16, 13, 18, 21, 17, 12, 19, \ldots, \phi(v_t)+\phi(v_{t-1})\}$ where $\phi(v_t) + \phi(v_{t-1}) = 8 + \frac{t-2}{2} + 13 + \frac{(t-1)-5}{2} = 17 + t$.

The set of edge weights can be reorder as a set of consecutive integers such as $\{2, 3, \ldots, t+17\}$. As this set contains distinct integers, then the map ϕ is the required k-labelling, which ends the proof. □

Example 3.12. Here is the labelling to state $es(L_{6,9})$.

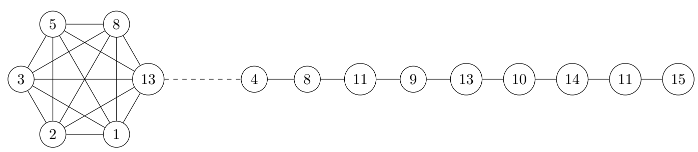

Figure 10: $es(L_{6,9}) = 15$

As it is clear, for each n, $3 \leq n \leq 6$ we found a way to label the vertices of each graph, so we end this section by the following open problem:

Open problem: State the edge irregularity strength for $L_{n,t}$ for any $t \geq 7$.

4. Modular edge irregularity strength of some Lollipop graphs

A *modular edge irregular strength* is a k-labelling $\phi : V(G) \longrightarrow \{1, 2, \ldots, k\}$ such that for any pair of different edges, the modular edge weights of them are distinct, and it is denoted by $mes(G)$. By *modular edge weight* is meant that the remainder upon dividing the edge weight by modulo of the size of the graph; that is by $|E(G)|$. If that labelling does not exist, then $mes(G) = \infty$.

The notion of the modular edge irregularity strength is recent, and was defined by Bača et al. in [5], and then it is improved by Koam et al. in [10]. The modular edge irregularity strength for some types of graphs as fan graphs and wheel graphs have been stated in [8], and for paths, cycles, caterpillar graphs, friendship graphs and n-sun graphs also have been evaluated in [10].

In this section, we state the modular edge irregularity strength for the above lollipop graphs, but we first include some of the needed results.

By its definition, one can see that the modular edge irregularity strength implies irregularity strength. That means for a simple graph G we have $es(G) \leq mes(G)$, but the converse is not true. The following theorem is of our importance.

Theorem 4.1. *Let G be a simple graph with $es(G) = k$. If edge weights under a corresponding edge irregular k-labelling constitute a set of consecutive integers, then $es(G) = mes(G) = k$.* □

Using this nice result, we have the following corollaries of theorems 3.3, 3.7, 3.9 and 3.11. respectively.

Corollary 4.2. *Let $G = L_{3,t}$ be the lollipop graph of $3 + t$ vertices. Then $mes(G) = \left\lceil \frac{3+t}{2} \right\rceil + 1$.* □

Corollary 4.3. *Let $G = L_{4,t}$ be the lollipop graph of $4 + t$ vertices. Then $mes(L_{4,t}) = 5 + \left\lfloor \frac{t}{2} \right\rfloor$.* □

Corollary 4.4. *Let $G = L_{5,t}$ be the lollipop graph of $5 + t$ vertices. Then $es(G) = 8$ for $t = 1$, and $mes(G) = 8 + \left\lfloor \frac{t+2}{2} \right\rfloor$ for $t \geq 2$.* □

Corollary 4.5. *Let $G = L_{6,t}$ be the lollipop graph of $6+t$ vertices. Then*

$$mes(G) = \begin{cases} 13, & \text{if } 1 \leq t \leq 6 \\ 13 + \left\lfloor \frac{t-6}{2} \right\rfloor, & \text{if t} > 6 \end{cases} \qquad \square$$

Example 4.6. Let us consider $L_{6,6}$ in figure 9. The set of consecutive edge weights is $3, 4, \dots, 23$. As $|E(L_{6,6})| = 21$, we divide the edge weights in the previous set by modulo 21, so we get the set of modular edge weights $\{3, 4, \dots, 20, 0, 1, 2\}$ respectively, which can be written as $\{0, 1, \dots, 20\}$.

5. Conclusion

In this paper, we discussed the edge irregularity strength of graphs that are joined to paths in some way; namely the fan graphs F_n, where we extended a result for $es(F_n)$ where $2 \leq n \leq 6$ given in [8] to $n \geq 2$. Also we stated the edge irregularity strength for some lollipop graphs $L_{n,t}$ for $n = 3, 4, 5, 6$. Finally we used our results about $es(L_{n,t})$ to state the modular edge irregularity strength for also the lollipop graphs $L_{n,t}$, $n = 3, 4, 5, 6$.

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Department of Mathematics
Tafila Technical University
Tafila
Jordan
E-mail: aalmazaydeh@ttu.edu.jo

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